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M a t h e m a t i c s

The inverse Sturm-Liouville problem with fixed boundary conditions

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It is well known the necessary and sufficient conditions for two sequences $\{\mu_n\}_{n=0}^{\infty} = \left\{\lambda_n^2\right\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ to be the spectral data for a certain Sturm-Liouville problem. We add to these conditions else two and they become necessary and sufficient conditions for $\left\{\lambda_n^2\right\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ to be the spectral data for a Sturm-Liouville problem with in advance given fixed boundary conditions.

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1. Introduction and statements of the results. Let us denote by $L(q,\alpha,\beta)$ the Sturm-Liouville boundary value problem

$$\ell y \equiv -y'' + q(x)y = \mu y, \quad x \in (0,\pi),\ \mu \in \mathbb{C}, \tag{1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad \alpha \in (0,\pi], \tag{2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \quad \beta \in [0,\pi), \tag{3}$$

where q is a real-valued, summable on $[0,\pi]$ function (we write $q \in L^1_{\mathbb{R}}[0,\pi]$). By $L(q,\alpha,\beta)$ we also denote the self-adjoint operator, generated by the problem (1)-(3) (see [1]). It is known, that under these conditions the spectra of the operator $L(q,\alpha,\beta)$ is discrete and consists of real, simple eigenvalues ([1]), which we denote by $\mu_n = \mu_n(q,\alpha,\beta) = \lambda_n^2(q,\alpha,\beta)$, $n = 0,1,2,\ldots$, emphasizing the dependence of μ_n on q, α and β. We assume that eigenvalues enumerated in the increasing order, i.e.

$$\mu_0(q,\alpha,\beta) < \mu_1(q,\alpha,\beta) < \ldots < \mu_n(q,\alpha,\beta) < \ldots.$$

Let $\varphi(x,\mu,\alpha,q)$ and $\psi(x,\mu,\beta,q)$ are the solutions of the equation (1), which satisfy the initial conditions

$$\varphi(0,\mu,\alpha,q) = \sin\alpha, \ \ \varphi'(0,\mu,\alpha,q) = -\cos\alpha,$$

$$\psi(\pi,\mu,\beta,q) = \sin\beta, \ \ \psi'(\pi,\mu,\beta,q) = -\cos\beta,$$

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correspondingly. The eigenvalues $\mu_n = \mu_n(q,\alpha,\beta)$, $n = 0,1,2,\ldots$, of $L(q,\alpha,\beta)$ are the solutions of the equation

$$\Phi(\mu) = \Phi(\mu,\alpha,\beta) \stackrel{def}{=} \varphi(\pi,\mu,\alpha)\cos\beta + \varphi'(\pi,\mu,\alpha)\sin\beta = 0,$$

or the equation

$$\Psi(\mu) = \Psi(\mu,\alpha,\beta) \stackrel{def}{=} \psi(0,\mu,\beta)\cos\alpha + \psi'(0,\mu,\beta)\sin\alpha = 0.$$

According to the well-known Liouville formula, the wronskian $W(x) = W(x,\varphi,\psi) = \varphi\cdot\psi' - \varphi'\psi$ of the solutions φ and ψ is constant. It follows that $W(0) = W(\pi)$ and, consequently $\Psi(\mu,\alpha,\beta) = -\Phi(\mu,\alpha,\beta)$. It is easy to see that the functions $\varphi_n(x) = \varphi(x,\mu_n,\alpha)$ and $\psi_n(x) = \psi(x,\mu_n,\beta)$, $n = 0,1,2,\ldots$, are the eigenfunctions, corresponding to the eigenvalue μ_n. Since all eigenvalues are simple,there exist constants $c_n = c_n(q,\alpha,\beta)$, $n = 0,1,2,\ldots$, such that

$$\varphi(x,\mu_n) = c_n\cdot\psi(x,\mu_n). \tag{4}$$

The squares of the L^2-norm of these eigenfunctions:

$$a_n = a_n(q,\alpha,\beta) = \int_0^\pi |\varphi_n(x)|^2 dx,\ n = 0,1,2,\ldots,$$

$$b_n = b_n(q,\alpha,\beta) = \int_0^\pi |\psi_n(x)|^2 dx,\ n = 0,1,2,\ldots$$

are called norming constants.

The sequences $\left\{\lambda_n^2\right\}_{n=0}^\infty$, $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ are called spectral data of problem $L(q,\alpha,\beta)$.

In this paper we consider the case $\alpha,\beta\in(0,\pi)$, i.e. we assume that $\sin\alpha\neq 0$ and $\cos\beta\neq 0$. In this case we consider the solution $\tilde{\varphi}(x,\mu,\alpha,q) := \frac{\varphi(x,\mu,\alpha,q)}{\sin\alpha}$ of the equation (1) which has the initial values

$$\tilde{\varphi}(0,\mu,\alpha,q) = 1,\ \tilde{\varphi}'(0,\mu,\alpha,q) = -\cot\alpha,$$

and also we consider the solution $\tilde{\psi}(x,\mu,\beta,q) := \frac{\psi(x,\mu,\beta,q)}{\sin\beta}$. Of course, the functions $\tilde{\varphi}_n(x) := \tilde{\varphi}(x,\mu_n,\alpha,q)$ and $\tilde{\psi}_n(x) := \tilde{\psi}(x,\mu_n,\alpha,q)$, $n = 0,1,2,\ldots$, are the eigenfunctions, corresponding to the eigenvalue μ_n. It follows from (4) that for norming constants $\tilde{a}_n := \|\tilde{\varphi}_n\|^2 = \frac{a_n}{\sin^2\alpha}$ $\tilde{b}_n := \|\tilde{\psi}_n\|^2 = \frac{b_n}{\sin^2\beta}$ the following connections

$$\tilde{b}_n = \frac{b_n}{\sin^2\beta} = \frac{a_n}{c_n^2\sin^2\beta} = \frac{\tilde{a}_n\sin^2\alpha}{c_n^2\sin^2\beta} \tag{5}$$

are held.

In this paper we will prove the following assertions.

Theorem 1. Let $q\in L^2_{\mathbb{R}}[0,\pi]$ and $\alpha,\beta\in(0,\pi)$. Then for norming constants $\tilde{a}_n$ and $\tilde{b}_n$ the following connections hold:

$$\frac{1}{\tilde{a}_0} - \frac{1}{\pi} + \sum_{n=1}^\infty\left(\frac{1}{\tilde{a}_n} - \frac{2}{\pi}\right) = \cot\alpha, \tag{6}$$

$$\frac{1}{\tilde{b}_0} - \frac{1}{\pi} + \sum_{n=1}^\infty\left(\frac{1}{\tilde{b}_n} - \frac{2}{\pi}\right) = -\cot\beta. \tag{7}$$

The inverse problem by "spectral function" (see [2, 5–9]) is in the reconstruction the problem $L(q,\alpha,\beta)$ by spectra $\{\mu_n\}_{n=0}^{\infty}$ and norming constants $\{\tilde{a}_n\}_{n=0}^{\infty}$ (or $\{\tilde{b}_n\}_{n=0}^{\infty}$).

Theorem 2. For real increasing sequence $\{\mu_n\}_{n=0}^{\infty}$ and positive sequence $\{\tilde{a}_n\}_{n=0}^{\infty}$ to be spectral data for boundary value problem $L(q,\alpha,\beta)$ with $q \in L^2_{\mathbb{R}}[0,\pi]$ and in advance fixed $\alpha,\beta \in (0,\pi)$ is necessary and sufficient that the following relations hold:

$$1)\ \lambda_n = \sqrt{\mu_n} = n + \frac{\omega}{\pi n} + \frac{\kappa_n}{n},\ \omega = const, \{\kappa_n\}_{n=0}^{\infty} \in l^2, \tag{8}$$

$$2)\ \tilde{a}_n = \frac{\pi}{2} + r_n,\ r_n = O\left(\frac{1}{n^2}\right), \tag{9}$$

$$3)\ \frac{1}{\tilde{a}_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty}\left(\frac{1}{\tilde{a}_n} - \frac{2}{\pi}\right) = \cot\alpha, \tag{10}$$

$$4)\ \frac{\tilde{a}_0}{\pi^2 \cdot \left(\prod_{k=1}^{\infty} \frac{\mu_k-\mu_0}{k^2}\right)^2} - \frac{1}{\pi} + \sum_{n=1}^{\infty}\left(\frac{\tilde{a}_n n^4}{\pi^2 \cdot [\mu_0-\mu_n]^2 \cdot \left(\prod_{k=1,k\neq n}^{\infty} \frac{\mu_k-\mu_n}{k^2}\right)^2} - \frac{2}{\pi}\right) = -\cot\beta. \tag{11}$$

■

2. The proof of the Theorem 1. For the solution $\tilde{\varphi}$ is well known the representation (see [2, 3, 9])

$$\tilde{\varphi}(x,\lambda,\alpha,q) = \cos\lambda x + \int_0^x G(x,t)\cos\lambda t dt, \tag{12}$$

where about the kernel $G(x,t)$ we know (in particular) that

$$G(x,x) = -\cot\alpha + \frac{1}{2}\int_0^x q(s)ds. \tag{13}$$

Besides, it is known that $G(x,t)$ satisfies to the Gelfand-Levitan integral equation

$$G(x,t) + F(x,t) + \int_0^x G(x,s)F(s,t)ds = 0,\ \ 0 \le t \le x, \tag{14}$$

where the function $F(x,t)$ is defined by formula (see for the details [9]

$$F(x,t) = \sum_{n=0}^{\infty}\left(\frac{\cos\lambda_n x\cos\lambda_n t}{\tilde{a}_n} - \frac{\cos nx\cos nt}{a_n^0}\right) \tag{15}$$

where $a_0^0 = \pi$ and $a_n^0 = \frac{\pi}{2}$ for $n = 1,2,\ldots$. It is easy follows from (13)-(15) that

$$G(0,0) = -F(0,0) = -\sum_{n=0}^{\infty}\left(\frac{1}{\tilde{a}_n} - \frac{1}{a_n^0}\right) =$$

$$= -\left(\frac{1}{\tilde{a}_0} - \frac{1}{\pi}\right) - \sum_{n=1}^{\infty}\left(\frac{1}{\tilde{a}_n} - \frac{2}{\pi}\right) = -\cot\alpha. \tag{16}$$

Thus, (6) is proved.

Let us now consider the functions ($n = 0,1,2,\ldots$)

$$p(x,\mu_n) = \frac{\varphi(\pi - x,\mu_n,\alpha,q)}{\varphi(\pi,\mu_n,\alpha,q)} = \frac{\varphi(\pi - x,\mu_n)}{\varphi(\pi,\mu_n)}. \tag{17}$$

Since $\varphi(x,\mu,\alpha,q)$ satisfies to the equation (1) and

$$p'(x,\mu_n) = -\frac{\varphi'(\pi - x,\mu_n)}{\varphi(\pi,\mu_n)}, \quad p''(x,\mu_n) = \frac{\varphi''(\pi - x,\mu_n)}{\varphi(\pi,\mu_n)},$$

we can see, that $p(x,\mu_n)$ satisfy to the equation

$$-p''(x,\mu_n) + q(\pi - x)p(x,\mu_n) = \mu_n p(x,\mu_n)$$

and initial conditions

$$p(0,\mu_n) = 1, \quad p'(0,\mu_n) = -\frac{\varphi'(\pi,\mu_n)}{\varphi(\pi,\mu_n)} = -(-\cot\beta) = \cot\beta = -\cot(\pi - \beta). \tag{18}$$

Also we have

$$p(\pi,\mu_n) = \frac{\varphi(0,\mu_n)}{\varphi(\pi,\mu_n)} = \frac{\sin\alpha}{\varphi(\pi,\mu_n)} = \frac{\sin(\pi-\alpha)}{\varphi(\pi,\mu_n)},$$

$$p'(\pi,\mu_n) = -\frac{\varphi'(0,\mu_n)}{\varphi(\pi,\mu_n)} = -\frac{-\cos\alpha}{\varphi(\pi,\mu_n)} = \frac{-\cos(\pi-\alpha)}{\varphi(\pi,\mu_n)}.$$

It follows, that $p_n(x) := p(x,\mu_n)$ satisfy to the boundary condition

$$p_n(\pi)\cos(\pi-\alpha) + p_n'(\pi)\sin(\pi-\alpha) = 0, \ n = 0,1,2,\ldots.$$

Let us denote $q^*(x) := q(\pi - x)$. Since $\mu_n(q^*,\pi-\beta,\pi-\alpha) = \mu_n(q,\alpha,\beta)$ (it is easy to prove and is well known, see, for example [6]), it follows, that $p_n(x), \ n = 0,1,2,\ldots,$ are the eigenfunctions of problem $L(q^*,\pi-\beta,\pi-\alpha)$, which have the initial conditions (18), i.e. $p_n(x) = \tilde{\varphi}(x,\mu_n,\pi-\beta,q^*), \ n = 0,1,2,\ldots.$

Thus, as in (16), for norming constants $\hat{a}_n = \|p(\cdot,\mu_n)\|^2$ must be true the connection

$$\left(\frac{1}{\hat{a}_0} - \frac{1}{\pi}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{\hat{a}_n} - \frac{2}{\pi}\right) = \cot(\pi-\beta) = -\cot\beta. \tag{19}$$

On the other hand, for norming constants $\hat{a}_n$,according to the (4), (5) and (17), we have

$$\hat{a}_n = \int_0^\pi p^2(x,\mu_n)dx = \int_0^\pi \frac{\varphi^2(\pi-x,\mu_n)}{\varphi^2(\pi,\mu_n)}dx =$$

$$= -\frac{1}{\varphi^2(\pi,\mu_n)}\int_\pi^0 \varphi^2(s,\mu_n)ds = \frac{1}{\varphi^2(\pi,\mu_n)}\int_0^\pi \varphi^2(s,\mu_n)ds =$$

$$= \frac{a_n(q,\alpha,\beta)}{\varphi^2(\pi,\mu_n)} = \frac{\tilde{a}_n\sin^2\alpha}{c_n^2\sin^2\beta} = \tilde{b}_n.$$

Therefore, we can rewrite (19) in the form

$$\left(\frac{1}{\tilde{b}_0} - \frac{1}{\pi}\right) - \sum_{n=1}^{\infty}\left(\frac{1}{\tilde{b}_n} - \frac{2}{\pi}\right) = -\cot(\pi-\beta) = \cot\beta.$$

Thus, (7) is true and Theorem 1 is proved.

3. The proof of the Theorem 2. For μ_n we have proved the following asymptotic formula [10]

$$\mu_n(q,\alpha,\beta) = [n+\delta_n(\alpha,\beta)]^2 + \frac{1}{\pi}\int_0^\pi q(t)dt + r_n(q,\alpha,\beta), \tag{20}$$

where δ_n is the solution of the equation

$$\delta_n(\alpha,\beta) = \frac{1}{\pi}\arccos\frac{\cos\alpha}{\sqrt{[n+\delta_n(\alpha,\beta)]^2\sin^2\alpha+\cos^2\alpha}} - \\ - \frac{1}{\pi}\arccos\frac{\cos\beta}{\sqrt{[n+\delta_n(\alpha,\beta)]^2\sin^2\beta+\cos^2\beta}} \tag{21}$$

and $r_n(q,\alpha,\beta) = o(1)$, when $n\to\infty$, uniformly in $\alpha,\beta\in[0,\pi]$ and q from any bounded subset of $L^1_{\mathbb{R}}[0,\pi]$ (we will write $q\in BL^1_{\mathbb{R}}[0,\pi]$). It follows from (21) (see [10] for details), that if $\sin\alpha\neq 0, \sin\beta\neq 0$, $(\alpha,\beta\in(0,\pi))$, then

$$\delta_n(\alpha,\beta) = \frac{\cot\beta-\cot\alpha}{\pi n} + O\left(\frac{1}{n^2}\right). \tag{22}$$

It follows from (20) that

$$\lambda_n = \sqrt{\mu_n} = n+\delta_n(\alpha,\beta) + \frac{[q]}{2\,[n+\delta_n(\alpha,\beta)]} + l_n + O\left(\frac{1}{n^2}\right), \tag{23}$$

where $l_n = \frac{1}{\pi[n+\delta_n(\alpha,\beta)]}\int_0^\pi q(x)\cos 2\lambda_n x dx = o\left(\frac{1}{n}\right)$ and $[q] = \frac{1}{\pi}\int_0^\pi q(t)dt$.

In case of $q\in L^2_{\mathbb{R}}[0,\pi]$ and $\alpha,\beta\in(0,\pi)$ it follows from (22) and (23) that we can rewrite (23) in the form

$$\lambda_n = n + \frac{\omega}{n} + \frac{\omega_n}{n}, \tag{24}$$

where ω = const = $\frac{\cot\beta-\cot\alpha+\frac{\pi}{2}[q]}{\pi}$ and $\{\omega_n\}_{n=0}^\infty\in l^2$, i.e. $\sum_{n=1}^\infty|\omega_n|^2<\infty$.

For norming constants the following asymptotic formulae hold (when $n\to\infty$)(see [11])

$$a_n(q,\alpha,\beta) = \frac{\pi}{2}\left[1+O\left(\frac{1}{n^2}\right)\right]\sin^2\alpha + \frac{\pi\cos^2\alpha}{2[n+\delta_n(\alpha,\beta)]^2}\left[1+O\left(\frac{1}{n^2}\right)\right] \tag{25}$$

$$b_n(q,\alpha,\beta) = \frac{\pi}{2}\left[1+O\left(\frac{1}{n^2}\right)\right]\sin^2\beta + \frac{\pi\cos^2\beta}{2[n+\delta_n(\alpha,\beta)]^2}\left[1+O\left(\frac{1}{n^2}\right)\right] \tag{26}$$

It is follows that for $\tilde{a}_n = \frac{a_n}{\sin^2\alpha}$ the relation (9) is held.

In [9] there is a proof of such assertion:

T h e o r e m 3. For real numbers $\left\{\lambda_n^2\right\}_{n=0}^\infty$ and $\{\tilde{a}_n\}_{n=0}^\infty$ to be the spectral data for a certain boundary value problem $L(q,\alpha,\beta)$ with $q\in L^2_{\mathbb{R}}[0,\pi]$, it is necessary and sufficient that the relations (24) and (9) are held.

It is known that the specification of the spectra $\{\mu_n(q,\alpha,\beta)\}_{n=0}^{\infty}$ uniquely determines the characteristic function $\Phi(\mu)$ (see [12], Lemma 2.2; see also [6], Lemma 1), and also its derivative $\frac{\partial\Phi(\mu)}{\partial\mu} = \dot{\Phi}(\mu)$ (see [12], lemma2.3). In particular, if $\alpha,\beta \in (0,\pi)$ the following formulae are held:

$$\dot{\Phi}(\mu_0) = -\pi \sin\alpha \sin\beta \prod_{k=1}^{\infty} \frac{\mu_k - \mu_0}{k^2} \tag{27}$$

and (if $n \neq 0$, i.e. $n = 1,2,\ldots$)

$$\dot{\Phi}(\mu_n) = -\frac{\pi}{n^2}\left[\mu_0 - \mu_n\right] \sin\alpha \sin\beta \prod_{k=1,k\neq n}^{\infty} \frac{\mu_k - \mu_n}{k^2}. \tag{28}$$

On the other hand, it is easy to prove the relation (see [12], formula (2.16) in Lemma 2.2 and see [6], Lemma 1)

$$a_n = -c_n \cdot \dot{\Phi}(\mu_n). \tag{29}$$

To take into account the connections (5) and (27)-(29) we can find formulae for $\frac{1}{\tilde{b}_0}$ and $\frac{1}{\tilde{b}_n}, n = 1,2,\ldots$:

$$\frac{1}{\tilde{b}_0} = \frac{\tilde{a}_0}{\pi^2 \cdot \left(\prod_{k=1}^{\infty} \frac{\mu_k - \mu_n}{k^2}\right)^2}, \tag{30}$$

$$\frac{1}{\tilde{b}_n} = \frac{\tilde{a}_n n^4}{\pi^2 \cdot [\mu_0 - \mu_n]^2 \cdot \left(\prod_{k=1,k\neq n}^{\infty} \frac{\mu_k - \mu_n}{k^2}\right)^2}. \tag{31}$$

So, we can change the second assertion in Theorem 1 by following

$$\frac{\tilde{a}_0}{\pi^2 \cdot \left(\prod_{k=1}^{\infty} \frac{\mu_k - \mu_n}{k^2}\right)^2} - \frac{1}{\pi} +$$

$$+\sum_{n=1}^{\infty}\left(\frac{\tilde{a}_n n^4}{\pi^2 \cdot [\mu_0 - \mu_n]^2 \cdot \left(\prod_{k=1,k\neq n}^{\infty} \frac{\mu_k - \mu_n}{k^2}\right)^2} - \frac{2}{\pi}\right) = -\cot\beta.$$

Thus we have a real sequence $\{\mu_n\}_{n=0}^{\infty} = \left\{\lambda_n^2\right\}_{n=0}^{\infty}$, which has the asymptotic representation (24) and a positive sequence $\{\tilde{a}_n\}_{n=0}^{\infty}$, which has the asymptotic representation (9). Then, according to the Theorem 3, there exist a function $q \in L^2_{\mathbb{R}}[0,\pi]$ and some constants $\tilde{\alpha}, \tilde{\beta} \in (0,\pi)$ such that λ_n^2, $n = 0,1,2,\ldots$, are the eigenvalues and $\tilde{a}_n$, $n = 0,1,2,\ldots$, are norming constants of the Sturm-Liouville problem $L(q,\tilde{\alpha},\tilde{\beta})$.

The function $q(x)$ and constants α,β are obtained on the way of solving the inverse problem by Gel'fand-Levitan method. The algorithm of that method is: at first we define the function $F(x,t)$ by formula (15) (note that this function is defined by $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\tilde{a}_n\}_{n=0}^{\infty}$ uniquely). And we will consider the integral equation (14), where $G(x,\cdot)$ is unknown function. It is proved that provided (9) and (24) the integral equation (14) has a unique solution $G(x,t)$. And with function $G(x,t)$ is constructed a function

$$\tilde{\varphi}(x,\lambda) = \cos\lambda x + \int_0^x G(x,t)\cos\lambda t\,dt, \tag{32}$$

defined for all $\lambda \in \mathbb{C}$. It is proved (see, for example, [9]) that

$$-\tilde{\varphi}''(x,\lambda^2)+\left(2\frac{d}{dx}G(x,x)\right)\tilde{\varphi}(x,\lambda^2)=\lambda^2\tilde{\varphi}(x,\lambda^2), \tag{33}$$

almost everywhere on $(0,\pi)$,

$$\tilde{\varphi}(0,\lambda^2)=1,$$
$$\tilde{\varphi}'(0,\lambda^2)=G(0,0).$$

If we state the condition

$$G(0,0)=-\cot\alpha, \tag{34}$$

then the solution (32) of the equation (33) will satisfy the boundary condition (2)

$$\tilde{\varphi}(0,\lambda^2)\cos\alpha+\tilde{\varphi}'(0,\lambda^2)\sin\alpha=0$$

for all $\lambda \in \mathbb{C}$. Since from (14) follows that $G(0,0)=-F(0,0)$ and from (15) follows that $F(0,0)=-\sum_{n=0}^{\infty}\left(\frac{1}{\tilde{a}_n}-\frac{1}{a_n^0}\right)$, then the condition (34) can be represented as

$$\sum_{n=0}^{\infty}\left(\frac{1}{\tilde{a}_n}-\frac{1}{a_n^0}\right)=\cot\alpha,$$

which is put on the sequence $\{\tilde{a}_n\}_{n=0}^{\infty}$.

It is also proved that the relation $\frac{\tilde{\varphi}_n'(\pi)}{\tilde{\varphi}_n(\pi)}=\frac{\tilde{\varphi}'(\pi,\lambda_n^2)}{\tilde{\varphi}(\pi,\lambda_n^2)}$ is a constant (i.e. does not depend on n), which we will denote by $-\cot\tilde{\beta}$. So the functions $\tilde{\varphi}(x,\lambda_n^2), n=0,1,2,\ldots$, are the eigenfunctions of the problem $L(q,\alpha,\tilde{\beta})$, where $q(x)=2\frac{d}{dx}G(x,x)$, α is in advance given α and we want $\tilde{\beta}$ to be equals β.

But we know, from the Theorem 1, that for problem $L(q,\alpha,\tilde{\beta})$ is true the equality

$$\sum_{n=1}^{\infty}\left(\frac{1}{\tilde{b}_n}-\frac{1}{\tilde{b}_n^0}\right)=-\cot\tilde{\beta}$$

and also are true the connections (30) and (31). Thus if we state the condition (11), then $\tilde{\beta}=\beta$. Theorem 2 is proved.

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